Math 432: Set Theory and Topology HOMEWORK 3 Due date: Feb 9 (Thu)

Reflections. Write a short essay (no more than 3/4 of a page) surveying what you have learnt so far in this course. Elaborate on equivalence relations and orderings. Pick a definition, a statement (e.g. theorem), and a proof that you liked best (i.e. *maximal* elements in your preference-order, don't have to be *maximum*) and explain your choice.

Exercises from Kaplansky's book.

Definition. A set A is called *countably infinite* if there is a bijection $\mathbb{N} \xrightarrow{\sim} A$. Call A *countable* if it is finite or countably infinite.

Sec 3.1: 2, 3, 6, 7, 8

Other (mandatory) **exercises.**

1. Determine which pairs of sets are isomorphic as ordered sets with their usual ordering $<$. Prove your answers.

(a) $\mathbb{N}$ and $\left\{-\frac{1}{n} : n \in \mathbb{N} - \{0\}\right\}$

(b) $\mathbb{Z}$ and $\left\{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\right\} \cup \{0\}$

(c) $\mathbb{R}$ and $(0, 1)$

(d) $\mathbb{Q}$ and $[0, 1) \cap \mathbb{Q}$

(e) $(0, 2)$ and $(0, 1) \cup (1, 2)$

(f) $(0, 2)$ and $(0, 1) \cup [2, 3)$.

2. (a) Let $(A, <)$ be a well-ordering and let $f : A \to A$ be an *order-homomorphism*, i.e.

$$a_0 < a_1 \implies f(a_0) < f(a_1)$$

for all $a_0, a_1 \in A$. Prove that f *progressive*, i.e. $a \leq f(a)$ for all $a \in A$.

(b) Deduce directly from part (a) that $(A, <) \not\prec (A, <)$ for any well-ordering $(A, <)$.

Remark. We proved this statement in class as a corollary of the uniqueness lemma for isomorphisms witnessing $\preceq$. This exercise provides a more direct proof.

(c) Ordering $\mathbb{N}^2$ lexicographically, give an example of an order-homomorphism $f : \mathbb{N}^2 \to \mathbb{N}^2$ (other than the identity map) such that $f(n, m) = (n, m)$ for all $(n, m) \geq_{\text{lex}} (2, 0)$.